Explosive amplification of an electromagnetic field in a magnetized flow of accelerated-electron oscillators

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The possibility of the explosive amplification of electromagnetic field by an ensemble of harmonic oscillators (with arbitrary population of excited energy levels) is established. The amplification occurs if the frequency of oscillators is chirped under the action of an external force. The analysis is given for the magnetized flow of electron cyclotron oscillators that are accelerated by the quasistationary electric field along the quasihomogeneous magnetic field. It is shown analytically that in such a nonsteady flow the initial energy of coherent oscillations of gyrating electrons can be efficiently converted into the energy of field as a result of quasiadiabatic tuning of electromagnetic wave frequency to the (Doppler-shifted) electron gyrofrequency. The optimal regime of the flow acceleration is found, in which the almost complete energy transfer occurs during a few periods of plasma oscillations. The proposed mechanism of explosive (nonexponential) amplification principally differs from and can prevail over the known maser and parametric mechanisms of field amplification.

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I. INTRODUCTION

It is well known that, when the recoil effect is neglected, the necessary conditions for the amplification of electromagnetic field by an ensemble of oscillators are (i) inversion of energy level population, and (ii) nonequidistant spacing of energy levels [1–3]. In this case, due to the induced radiation of nonlinear oscillators, the instability becomes possible, leading to the simultaneous exponential growth of the high-frequency field and oscillation amplitudes. Besides the usual atomic masers and lasers, the typical example is the ensemble of electrons gyrating in the magnetic field. The nonequidistance of their energy levels (Landau levels) arises from the relativistic effect of velocity-dependent electron mass, and it is due to this effect that the gyrotron works [4–7]. In the nonrelativistic approximation that is employed below, the cyclotron oscillators are harmonic, i.e., linear, and they can give rise only to the induced absorption of the field, independently of the presence of population inversion.

However, the conclusion that an ensemble of *harmonic* oscillators cannot amplify the electromagnetic field is not applicable to the nonsteady situation, when, for example, the oscillators are accelerated by electric dc field or gravitational field, or the electron gyrofrequency changes in the nonsteady (or inhomogeneous) magnetic field. As will be shown below, in this situation the amplification of electromagnetic field is possible provided the initially coherent wave of phased electron oscillations is somehow created in the oscillator ensemble, e.g., by a preliminary coherent pumping or as a result of intrinsic instability.

The new mechanism of the field amplification takes place in the case of quasiadiabatic tuning of the oscillator frequency to the frequency of the electromagnetic wave. In this case, the quasiadiabatic change of the ratio of electric field amplitude to current density amplitude in the nonstationary self-consistent wave can be possible, in which the relatively small frequency shift can lead to an almost complete coherent transfer of the energy of oscillators into the electromagnetic field energy. The remarkable feature of the amplification process is *the explosive (nonexponential) growth* of the field amplitude. The nontrivial fact, which was the main motivation for the present paper, is the possibility to carry out this ‘‘explosive’’ energy transfer during a very short time as compared with characteristic times of all exponential instabilities (and relaxation processes) in a system ‘‘oscillators plus field.’’ (The rate of the field growth is terminated by violation of adiabaticity, i.e., by the linear coupling between electronic and electromagnetic degrees of freedom [8–13], see below.) To our knowledge, this important fact has not been studied so far, either in the general theory of oscillations and waves or in the applied electrodynamics of plasma and electronic systems.

It should be stressed that the explosive amplification is the *linear* effect in a nonsteady system, and has nothing to do with explosive nonlinear interaction, well known in plasma physics and electronics [7,14]. Besides, by considering only the quasiadiabatic acceleration of the electrons (or variation of their gyrofrequency), we exclude the evident possibility of the parametric amplification in the high-frequency external field that is used in the parametric electronic and optical generators and free-electron lasers [10,15–20].

The influence of dc electric or inhomogeneous magnetic fields on the interaction of electromagnetic perturbations with the cyclotron oscillator beam was studied before, mainly within the problem of maintaining wave-particle synchronism in order to increase the efficiency of generators [21–23]. The investigation was also made on a quasiadiabatic evolution of the steady state of electrons in a strong electromagnetic wave propagating along the magnetic field and dc electric field, the latter being treated as a perturbation [24]. In these papers, the externally driven shift of the oscillator frequency was not treated self-consistently, and was considered a minor correction factor providing the better efficiency.

In the present paper we point out the linear nonstationary

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amplification mechanism that owes its very existence to the accelerating electric dc field. The emphasis is made on the extremely high rate of the explosive amplification, inaccessible for the traditional mechanisms of the exponential amplification. Mathematically, the analysis of the highest amplification rate is reduced to the problem of the optimal control of the variable parameters in a system of linear differential equations. This problem is solved on the basis of approximate analytical solutions, which give the complete physical picture of the phenomenon.

The paper is organized as follows. The formulation of the basic nonstationary problem and the hydrodynamical analysis of the cyclotron waves in the accelerated electron beam are given in Sec. II. In Sec. III, the adiabatic solution is presented, and the general restrictions on the value of efficiency and the factor of field amplification are formulated. Section IV is devoted to the case of a constant acceleration of electrons. An exact nonadiabatic solution is derived; the limiting rate of the explosive amplification, its delay time, and the maximum amplification factor are found. In Sec. V, the general quasiadiabatic solution is found. The optimal regime of variable electron acceleration, minimizing the delay time of the explosive amplification to a few periods of plasma oscillations, is pointed out. In Sec. VI, the comparison with the known instabilities, first of all, the kinetic ones, is made and the amplification of an envelope of cyclotron waves is discussed. Some numerical estimates and the required parameters of an electron beam for the realistic conditions in the laboratory and space plasmas are given in Sec. VII. In the concluding Sec. VIII we discuss the general features of the effect that are independent of the model of harmonic oscillators.

II. GENERIC EXAMPLE OF THE CYCLOTRON OSCILLATORS: THE HYDRODYNAMICAL ANALYSIS OF *X*-MODE DISPERSION

As a "generating problem" we consider the one-dimensional electron flow in quasistatic electric and magnetic fields, which is assumed for simplicity to be homogeneous and parallel, $\mathbf{E_0}(t)\|\mathbf{B_0}(t)\|\mathbf{z^0}$. The electron beam is also considered homogeneous, with time-independent density, $N = \text{const}$, and the charge being compensated by a background of heavy positive ions (with background dielectric constant ε_0). Their regular motion with respect to the electron beam is unimportant for the mechanism of explosive amplification. Therefore, the nonelectromagnetic methods of electron acceleration are allowed, e.g., by the light pressure or by the gravitational field accelerating ions as well.

Consider the temporal evolution of the electromagnetic field in the *extraordinary mode* (*X* mode) near the resonance with electron gyrofrequency ω_B. Suppose that electrons are strongly magnetized and have a small dispersion of longitudinal velocities, $v_{T\|}$, and the electron-electron (ν_{ee}), electron-ion (ν_{ei}), and electron-neutral (ν_{en}) collision frequencies are also small as compared with the electron plasma frequency ω_L:

$$kv_{T\|}+\nu\ll\omega_L\ll\omega_B, \tag{1}$$

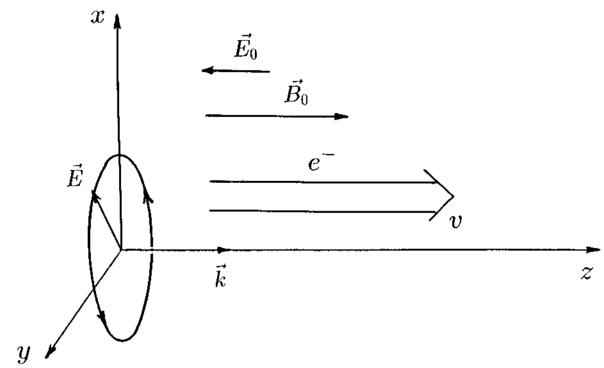

FIG. 1. Nonstationary *X* mode in one-dimensional flow of cyclotron oscillators, accelerated in the parallel $\mathbf{E_0}$, $\mathbf{B_0}$ fields. An electric field vector $\mathbf{E}$ in the mode is rotating in the direction of electrons in an external magnetic field $\mathbf{B_0}$.

where $\nu=\nu_{ee}+\nu_{ei}+\nu_{en}$, $\omega_L=4\pi e^2N/m$,, $\omega_B=eB_0/mc$, $-e$ and m are the charge and mass of the electron, and c the velocity of light in vacuum.

We shall restrict ourselves to the case of the longitudinal propagation of a transverse circularly polarized *X* mode (Fig. 1) with the wave vector $\mathbf{k}\|\mathbf{z^0}$, the density of electron current $\mathbf{j}(z,t)\perp\mathbf{z^0}$, and the electric field vector $\mathbf{E}(z,t)\perp\mathbf{z^0}$ lying in the (x,y) plane and rotating in the direction of electrons in an external magnetic field:

$$\partial^2\mathbf{E}/\partial t^2-c_0^2\partial^2\mathbf{E}/\partial z^2=-4\pi\varepsilon_0^{-1}\partial\mathbf{j}/\partial t, \tag{2}$$

$c_0\equiv c\varepsilon_0^{-1/2}$. Assuming the *homogeneous initial-value problem*, consider the quasimonochromatic plane wave, harmonic in space and with slowly varying complex amplitude, $\mathscr{E}(t)$:

$$\mathbf{E}(z,t)=(1/2)\mathscr{E}(t)\exp(-i\omega t+ikz)+\text{ c.c.} \tag{3}$$

Here $k\equiv k_z=\text{Re}(k)$ and the resonance condition will be used, $|\omega-\omega_B|\ll\omega_B$.

Suppose that at the initial moment of time, $t=0$, there existed one spatial harmonic of the eigencyclotron wave of a beam that was excited by a coherent pumping, and this wave consisted of a (strong) spatial harmonic of the current density $\mathbf{j}$ and the corresponding (weak) harmonic of the electric field (3). The more general case of a wave envelope in a beam with inhomogeneous density $N(z)$, in the inhomogeneous external fields $\mathbf{E_0}(z)$ and $\mathbf{B_0}(z)$, is discussed in the end of Sec. VI within spatiotemporal WKB approximation [25–27].

Our aim is to prove the possibility of and to find out the conditions for the explosive amplification of the self-consistent amplitudes of the electric field $\mathscr{E}(t)$ and the cyclotron current density $\mathscr{J}(t)$, the latter being introduced as in Eq. (3). It will be shown below that the effect is linear and nonrelativistic, and can be realized only out of the cyclotron line:

$$kv_{T\|}+\nu<|\tilde{\delta}|\ll\omega_B,\quad \tilde{\delta}=\omega-\omega_B-kv,$$
$$v=v_z(t)\ll c\sim c_0\simeq\omega_B/k\simeq\omega/k. \tag{4}$$

Therefore, we may use an approximate hydrodynamical description of the transverse current density $\mathbf{j}\equiv-eN\mathbf{v}_\perp$ of electron gyrooscillations in the beam:

$$d\mathbf{j}/dt \equiv \partial \mathbf{j}/\partial t + v(t)\partial \mathbf{j}/\partial z = (\omega_L^2/4\pi)\mathbf{E} - \omega_B(t)[\mathbf{j}\times\mathbf{B_0}/B_0]. \tag{5}$$

Neglecting the relativistic effects and the nonlinear Lorentz force from the high-frequency magnetic field of the wave, we may use Eq. (5) for arbitrary electron distribution over the Landau levels, i.e., over the transverse (chaotically distributed) velocities $u_\perp$. From Eqs. (2) and (5) within the resonance approximation (3) and (4), we obtain the truncated equations for the "circular" combinations of the field and current components, $\mathscr{E} = \mathscr{E}_x - i\mathscr{E}_y$ and $\mathscr{J} = \mathscr{J}_x - i\mathscr{J}_y$:

$$d\mathscr{E}/dt - i\delta\mathscr{E} = -2\pi\varepsilon_0^{-1}[(\omega_B + kv)/\omega]\mathscr{J},$$

$$d\mathscr{J}/dt - i\tilde{\delta}\,\mathscr{J} = (\omega_L^2/4\pi)\mathscr{E}. \tag{6}$$

Here the factor $(\omega_B + kv)/\omega \simeq 1$, and along with the Doppler detuning from the cyclotron resonance $\tilde{\delta}$ in Eq. (4), we introduce the detuning from the electromagnetic resonance:

$$\delta = (\omega^2 - k^2c_0^2 - \omega_L^2\varepsilon_0^{-1})/2\omega \simeq \omega - kc_0;\, |\delta| \ll \omega_B. \tag{7}$$

In the steady state or for the instantaneous values of parameters, Eqs. (6) give the dispersion equation and the relation between the field and current amplitudes in a normal wave:

$$\frac{\mathscr{E}}{\mathscr{J}} = -\frac{i\tilde{\delta}4\pi}{\omega_L^2} \equiv -\frac{i2\pi}{\varepsilon_0\delta}. \tag{8}$$

The solution of this well-known dispersion equation consists of two smooth curves,

$$\omega_{1,2}(k) = \omega_B + kv + \tilde{\Delta}/2 \pm \sqrt{\tilde{\Delta}^2/4 + \omega_L^2/2\varepsilon_0};$$

$$\tilde{\Delta} \equiv \tilde{\delta} - \delta = kc_0 - kv - \omega_B, \tag{9}$$

lying above and below the line $\omega = \omega_B + kv$ that corresponds to the *partial cyclotron wave*, and the line $\omega = (k^2c_0^2 + \omega_L^2/\varepsilon_0)^{1/2} \simeq kc_0$ that corresponds to the *partial electromagnetic wave*, see Fig. 2. Depending on the sign of the frequency shift between the partial electromagnetic and cyclotron waves, $\tilde{\Delta}(k)$, it can be convenient to rename the dispersion curves (9) by tearing them at the resonance point, $k_r = \omega_B/(c_0 - v)$, where $\tilde{\Delta} = 0$:

$$\omega_{c,\mathrm{em}}(k) = \omega_B + kv + \tilde{\delta}_{c,\,\mathrm{em}}(k); \tag{10}$$

$$\tilde{\delta}_{c,\mathrm{em}} = (1 \mp \sqrt{1 + 2\omega_L^2/\varepsilon_0\tilde{\Delta}^2})\tilde{\Delta}/2.$$

The resulting branches of *normal cyclotron and normal electromagnetic waves* are shifted from the frequency of the partial cyclotron wave by the value greater than and smaller than $\omega_L/\sqrt{2\varepsilon_0}$, respectively. Therefore, according to the ratio $\mathscr{E}/\mathscr{J}$ in Eq. (8), the normal cyclotron wave $\omega_c(k)$ is mainly presented by oscillations of the cyclotron current, and not by the electromagnetic field. On the contrary, the normal electromagnetic wave $\omega_{\mathrm{em}}(k)$ consists mainly of electromagnetic field oscillations. This follows directly from the electrodynamical expression for the wave energy density in the system "beam + field":

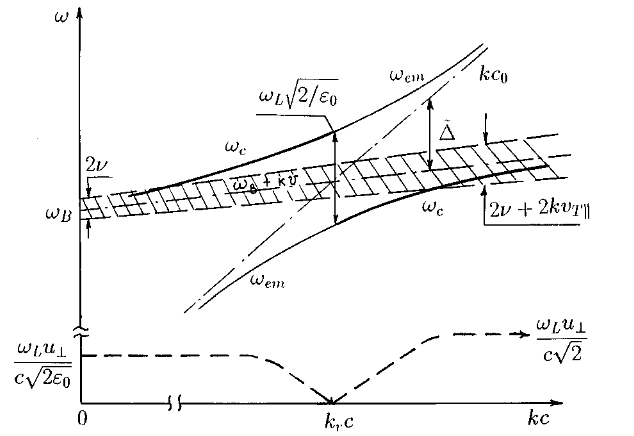

FIG. 2. The cyclotron curve $\omega_c(k)$ (thick solid line) and the electromagnetic curve $\omega_{\mathrm{em}}(k)$ (thin solid line) of the X mode for the longitudinal propagation $(\mathbf{k}\|\mathbf{B_0})$ in the flow of electrons with the same longitudinal velocity v. The cyclotron line with strong wave dissipation is hatched. Note that near the resonance the cyclotron curve lies outside this line if $|k - k_r| < \delta_r = \omega_L^2/2\varepsilon_0(c_0 - v)(k_r v_{T\|} + \nu)$. The dashed line shows the limiting growth rate.

$$W = |\mathscr{E}^2|\varepsilon_0/16\pi + |\mathscr{J}^2|\pi/2\omega_L^2$$
$$\equiv [2\varepsilon_0(\tilde{\delta}/\omega_L)^2 + 1](\pi/2\omega_L^2)|\mathscr{J}^2|. \tag{11}$$

It is important that in the cyclotron wave ω_c the electric field amplitude is inversely proportional to the frequency shift $\tilde{\Delta}$, and the field energy is small as compared with the energy of phased gyrorotation of electrons if the shift is large enough, $|\tilde{\Delta}| > \omega_L\sqrt{2/\varepsilon_0}$:

$$\left(\frac{\mathscr{E}}{\mathscr{J}}\right)_c = -\frac{i\tilde{\delta}_c 4\pi}{\omega_L^2} \simeq \frac{i2\pi}{\varepsilon_0\tilde{\Delta}}; \tag{12}$$

$$\tilde{\delta}_c \simeq -\frac{\omega_L^2}{2\varepsilon_0\tilde{\Delta}}.$$

Hence, even at the fixed wave energy $W_c =$ const, we can strongly increase the field amplitude $\mathscr{E}_c$ with very slight change of the current amplitude $\mathscr{J}_c$. To achieve this, one should diminish the frequency shift $\tilde{\Delta}$ from the large initial value, $|\tilde{\Delta}_0| \gg \omega_L\sqrt{2/\varepsilon_0}$, to the value of the order of plasma frequency, $\omega_L\sqrt{2/\varepsilon_0}$, by changing the longitudinal velocity of electrons $v(t)$, their gyrofrequency $\omega_B(t)$, or the velocity of light in the background medium $c_0(t)$. According to Eq. (12), the field amplification is of the explosive type, its rate being determined by the rate of frequency shifting, i.e., by the electron acceleration in our example:

$$d\tilde{\Delta}/dt = -ka; \tag{13}$$

$$a(t) = -(e/m)E_{0z}(t) \equiv dv/dt.$$

As shown schematically in Fig. 3, the mechanism of explosive amplification is connected with the shift of nonstationary dispersion branches $\omega_{c,\mathrm{em}}(k,t)$ with respect to the given spatial harmonic of the field, $k =$ const. Step 1 corresponds

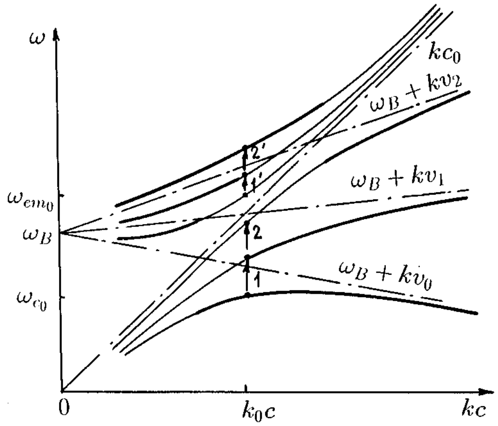

FIG. 3. The scheme illustrating the evolution of dispersion curves for the longitudinal ($\|\mathbf{B_0}$) acceleration of electrons from the initial velocity v_0 to the velocity v_1 and then to v_2. Thick (thin) solid line is the cyclotron (electromagnetic) branch. The arrows correspond to: (1) explosive amplification of the electric field with $\mathscr{J}_c$ = const and (2) decrease of the cyclotron current with $\mathscr{E}_c$ = const, when only the cyclotron wave with frequency $\omega_{c_0}(k_0)$ is initially present, ($1'$) explosive amplification of the cyclotron current with $\mathscr{E}_{\rm em}$ = const and ($2'$) decrease of the electric field with $\mathscr{J}_{\rm em}$ = const, when only the electromagnetic wave with frequency $\omega_{{\rm em}_0}(k_0)$ is initially present.

to the field amplification. At step $1'$ the field remains practically unchanged, while the cyclotron current is amplified.

III. ADIABATIC SOLUTION: EFFICIENCY AND THE MAXIMUM FACTOR OF THE EXPLOSIVE AMPLIFICATION

Now we turn from the qualitative considerations to the rigorous solution of the above-formulated problem. First, we eliminate the arbitrariness in the choice of the frequency ω in Eqs. (6) by redefining the complex amplitudes of electric field $\tilde{\mathscr{E}}$ and cyclotron current $\tilde{\mathscr{J}}$:

$$\left.\begin{matrix} f \\ g \end{matrix}\right\} = \left\{\begin{matrix} \sqrt{\varepsilon_0/16\pi}(\tilde{\mathscr{E}}_x - i\tilde{\mathscr{E}}_y) \\ \sqrt{\pi/2}\,\omega_L^{-1}(\tilde{\mathscr{J}}_x - i\tilde{\mathscr{J}}_y) \end{matrix}\right\} \times \exp\left(\frac{i}{2}\int_0^t \tilde{\Delta}(\tau)d\tau\right), \tag{14}$$

where the current and field amplitudes are defined using the Doppler-shifted frequency $\omega_B + kv$:

$$\left.\begin{matrix} \mathbf{E} \\ \mathbf{j} \end{matrix}\right\} = \frac{1}{2} \times \left\{\begin{matrix} \mathscr{E}(t) \\ \mathscr{J}(t) \end{matrix}\right\} \times \exp\left[-i\int_0^t [\omega_B(\tau) + kv(\tau)]d\tau + ikz\right] + \text{c.c.} \tag{15}$$

[cf. Eq. (3)]. Equations (6) now take the dimensionless form

$$f' + iqf = -g, \quad g' - iqg = f, \tag{16}$$

where we introduce the normalized frequency shift, which is a time-dependent function:

$$q(\theta) = \tilde{\Delta}\omega_L^{-1}\sqrt{\varepsilon_0/2}; \quad \theta = t\omega_L/\sqrt{2\varepsilon_0}. \tag{17}$$

The prime denotes differentiating over the dimensionless time θ. Equation (16) provides the energy conservation [cf. Eq. (11)]: $W = |f^2| + |g^2|$ = const. For the fixed wave number k, the field-to-current ratio (8) in two normal waves (9) is determined by the frequency shift q:

$$\frac{f}{g} \equiv \left(\frac{\omega_L\sqrt{2\varepsilon_0}}{4\pi}\right)\frac{\tilde{\mathscr{E}}}{\tilde{\mathscr{J}}} = \frac{i}{K_{1,2}}, \tag{18}$$

$$K_{1,2}(\theta) = q \mp \sqrt{q^2+1}.$$

Following Ref. [13], the general solution of Eqs. (16) can be expressed via new amplitudes:

$$\left.\begin{matrix} f \\ g \end{matrix}\right\} = \frac{F_1}{\sqrt{1+K_1^2}} \times \left\{\begin{matrix} 1 \\ -iK_1 \end{matrix}\right\} + \frac{F_2}{\sqrt{1+K_2^2}} \times \left\{\begin{matrix} 1 \\ -iK_2 \end{matrix}\right\}. \tag{19}$$

In the WKB approximation, only the phases of the amplitudes of normal waves may vary:

$$F_{1,2}(\theta) = F_{1,2}(0)\exp\left(\pm i\int_0^\theta \sqrt{q^2+1}\,d\theta'\right)$$
$$\equiv F_{1,2}(0)\exp\left(\pm i\int_0^t \sqrt{\tilde{\Delta}^2/4 + \omega_L^2/2\varepsilon_0}\,d\tau\right). \tag{20}$$

Consider the normal cyclotron wave, which has, by definition, $|K_c| = |q \mp \sqrt{q^2+1}| \geqslant 1$, i.e., either (i) $K_c = K_1$ for $q \leqslant 0$, or (ii) $K_c = K_2$ for $q \geqslant 0$. Suppose we have the second case, and there initially exists only the cyclotron wave $[F_1(0) = 0, F_2(0) \neq 0]$ with large positive value: $q(0) = q_0 \gg 1$. In this case, the slow adiabatic decrease of the shift (17) (by the acceleration of electrons along $\mathbf{B_0}$) leads to the displacement along the cyclotron branch, $\omega_c = \omega_2(k,t)$ (step 1 in Fig. 3), i.e., to the explosive amplification of the field:

$$|f(\theta)| = \frac{|F_2(0)|}{\sqrt{1+K_2^2}} \simeq \frac{|F_2(0)|}{2q(\theta)} \propto \frac{1}{\tilde{\Delta}(t)}. \tag{21}$$

The amplification terminates at $q \sim 1$; cf. Eq. (12). With further acceleration, we go smoothly through the resonance where the shift $q(\theta)$ changes sign and becomes negative and large. This domain corresponds to the electromagnetic branch, $\omega_2 = \omega_{\rm em}$, where the field amplitude cannot increase significantly due to $K_2^2 \lesssim 1$ in the solution of (19) and (20).

A similar situation takes place in the case (i) for the large negative initial shift, $-q_0 \gg 1$. The difference is that, when starting from the initial cyclotron wave $[F_1(0) \neq 0, F_2(0) = 0]$, the explosive amplification is obtained during the counter displacement along the branch $\omega_c = \omega_1(k,t)$ if the electron acceleration is directed opposite $\mathbf{B_0}$ (step $2'$ in Fig. 3).

Anyway, to obtain the explosive field amplification, the sign of the projection of electron acceleration (or deceleration) on the wave vector, ka, should coincide with the sign of the initial frequency shift, $\tilde{\Delta}_0 = kc_0 - kv_0 - \omega_B$; see Fig. 3. For $\mathbf{k}\uparrow\uparrow\mathbf{B_0}$ (codirected vectors) in Fig. 1, this amounts to choosing (i) $\mathbf{E_0}\uparrow\uparrow\mathbf{B_0}$ for $k < k_r$ and (ii) $\mathbf{E_0}\downarrow\uparrow\mathbf{B_0}$ (counterdirected vectors) for $k > k_r$. Under these conditions, the small change in the frequency of the electromagnetic field will be accompanied by the growth of its amplitude due to energy transfer from the electrons located in the decelerating phase of the field. This can be easily checked by averaging (over the random initial phases of gyrating electrons) the work done by the self-consistent field of a cyclotron wave:

$$A_\perp \equiv -e\left\langle \int_0^t (v_x E_x + v_y E_y)\, d\tau \right\rangle = (\varepsilon_0/8\pi)[|\tilde{\mathscr{E}}(0)|^2 - |\tilde{\mathscr{E}}(t)|^2] < 0. \tag{22}$$

In the adiabatic approximation, the growth of the *explosive amplification factor*,

$$\kappa \equiv \frac{|\tilde{\mathscr{E}}_{\rm out}|}{|\tilde{\mathscr{E}}_{\rm in}|} = \frac{(1+K_c^2)_{\rm in}^{1/2}}{(1+K_c^2)_{\rm out}^{1/2}}, \tag{23}$$

terminates when the vicinity of resonance is reached, where $|q_{\rm out}| \lesssim 1$. The maximum is

$$\kappa_m \simeq \sqrt{2}|q_0| = |\tilde{\Delta}_0| \sqrt{\varepsilon_0}/\omega_L . \tag{24}$$

Thus, it is desirable to increase the initial shift $|\tilde{\Delta}_0|$ and to decrease the plasma frequency, i.e., the electron density. However, at the fixed value of acceleration, this would prolong the time of the amplification process. Second, with decreasing plasma frequency the allowable values of electron acceleration [Eqs. (33), (40)] are also decreased. Third, an initial shift is bounded from above by the relaxation in the cyclotron line, $|\tilde{\Delta}_0| \lesssim \omega_L^2 [2\varepsilon_0(\nu + \omega_B v_{T\parallel}/c_0)]^{-1}$; see the inequality (4) and Fig. 2. Even in the ideal case of the absence of electron relaxation, the resonance approximation requires that $|\tilde{\Delta}_0| \lesssim \omega_B/2$. Therefore, we always have

$$\kappa_m < \kappa_\Delta \simeq \frac{\sqrt{\varepsilon_0}}{2}\left(\frac{\omega_L}{\varepsilon_0(\nu + \omega_B v_{T\parallel}/c_0)} + \frac{\omega_B}{\omega_L}\right). \tag{25}$$

Besides, one can show that the nonlinear bunching of electrons due to the magnetic field of the X mode does not violate the condition $|\tilde{\mathscr{E}}\tilde{\Delta}| \simeq |\tilde{\mathscr{E}}_0 \tilde{\Delta}_0|$, which we use to connect the initial ($0=$ in) and final (out) amplitudes, only if the initial shift is small enough:

$$\kappa_m \simeq \frac{|\tilde{\Delta}_0|}{\omega_L} \ll \kappa_0 \simeq \frac{(\omega_L/2\omega_B)^3 B_0^2}{4\pi W_0}. \tag{26}$$

Here we take $|\tilde{\Delta}_{\rm out}| \simeq \omega_L$ at the final time $\theta_{\rm out}$ and $\varepsilon_0 \simeq 1$ for simplicity, and also introduce the initial energy density of a cyclotron wave,

$$W_0 \simeq \pi \left|\frac{\varepsilon_0 \tilde{\mathscr{E}}_0 \tilde{\Delta}_0}{2\pi\omega_L}\right|^2 \sim \frac{|\tilde{\mathscr{E}}_{\rm out}|^2}{4\pi}. \tag{27}$$

Define the maximum efficiency η as a ratio of the maximum energy density of the electromagnetic field $W_0/2$, reached near the resonance after the explosive amplification, to the work required for the acceleration of electrons per unit volume, $NA_\parallel = Nm(v_{\rm out}^2 - v_0^2)/2$:

$$\eta \simeq \frac{W_0}{2NA_\parallel} = \frac{(\omega_B/\tilde{\Delta}_0)^2 W_0}{Nmc_0^2} \ll \frac{\varepsilon_0 \omega_L^2 \omega_B}{8|\tilde{\Delta}_0^3|}. \tag{28}$$

In the last inequality we take $v_0 = 0$ and the minimally possible initial shift, $|\tilde{\Delta}_0| \sim 2\omega_L\sqrt{2/\varepsilon_0}$, which still allows for the significant explosive amplification of the electromagnetic field. Thus, even with nonlinear limitation (26) taken into account in the inequality (28), the above-defined amplification efficiency may be much greater than unity. This does not create confusion because the work done by the dc electric field does not turn directly into the electromagnetic field energy. It is the initially stored energy of the coherent electron oscillations in a cyclotron wave that is transformed into the field energy. And, of course, the transformation factor of this stored initial energy cannot be greater than unity.

IV. LINEAR WAVE COUPLING IN THE CASE OF A CONSTANT ELECTRON ACCELERATION

The most interesting problem is the limiting rate of explosive amplification, since it is the unexpectedly high rate of the process that makes the effect nontrivial. To find the limits to the adiabatic approximation (20), i.e., the conditions for weak linear transformation of the normal waves (9), we can treat Eqs. (19) as the introduction of new variables, and write down the coupled equations for these complex amplitudes of the normal waves [13]:

$$F_1' + i\sqrt{q^2+1}F_1 = \frac{q' F_2}{2(q^2+1)},$$

$$F_2' - i\sqrt{q^2+1}F_2 = -\frac{q' F_1}{2(q^2+1)}. \tag{29}$$

It is clear that the transformation $F_1 \leftrightarrow F_2$, which terminates the explosive amplification of the field, increases with growing value of the derivative $q' \equiv \varepsilon_0 \omega_L^{-2} d\tilde{\Delta}/dt = -\varepsilon_0 \omega_L^{-2} ka$, and becomes most important near the resonance, where $|q| \sim 1$. Therefore, we should analyze the transition from the domain $|q_{\rm in}| \gg 1$ to the domain $|q_{\rm out}| \ll 1$.

The solutions to Eqs. (29) are expressed via the Weber functions [12] in the case of a constant electron acceleration (13), E_0 = const, when $q(\theta) = (\theta - \theta_{\rm out})q'$. The resulting transformation factor Q of the normal cyclotron wave to the normal electromagnetic one (10),

$$Q \equiv \frac{|F_{\rm em}^{\rm out}|^2}{|F_c^{\rm in}|^2} = \frac{|F_{\rm em}^{\rm out}|^2}{|F_{\rm em}^{\rm out}|^2 + |F_c^{\rm out}|^2}, \tag{30}$$

is shown in Fig. 4 for $|q_{\rm in}| \to \infty$, $|q_{\rm out}| \to 0$. The adiabatic approximation fails at $|q'| \simeq 2$:

$$Q \simeq \left(\frac{q'}{4}\right)^2 \ll 1 \quad \text{for } |q'| \ll 2,$$

$$Q \simeq \frac{1}{2} - \left(\frac{\pi}{8|q'|}\right)^{1/2} \quad \text{for } |q'| \gg 2. \tag{31}$$

For large $|q'|$ the explosive amplification is practically absent. The same result can be obtained, according to Refs. [8,9], from the solution to Eqs. (29) in the case of not only $-q_{\rm in} \to \mp\infty$, but also $q_{\rm out} \to \mp\infty$, when we go through the resonance along the electromagnetic wave curve; see Fig. 3.

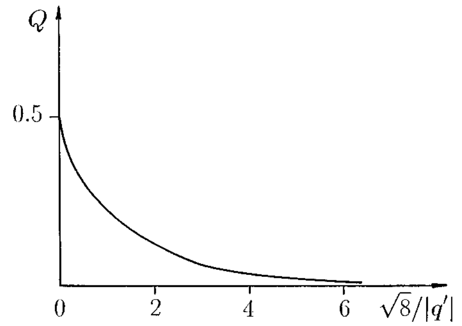

FIG. 4. The transformation factor Q [Eq. (30)] of the normal cyclotron wave to the normal electromagnetic wave as a function of $\sqrt{8}/|q'| = \sqrt{8} c_0 \omega_L^2 / \varepsilon_0 \omega_B |a|$, for $a = \text{const}$. Initially there exists one normal cyclotron wave far from the resonance $(|\tilde{\Delta}_0| \gg \omega_L \sqrt{2/\varepsilon_0})$.

In this case it is useful to express the solution as a transformation factor from one of the dispersion curves (9), $\omega_{1,2}(k,t)$, to another:

$$Q_{2\to 1} = |F_1^{\text{out}}|^2 / |F_2^{\text{in}}|^2 = \exp(-\pi/|q'|). \tag{32}$$

Thus, the results of Sec. III are valid only until the electron acceleration is less than

$$a_{\text{cr}} \simeq \frac{2\omega_L^2}{\varepsilon_0 k} \simeq \frac{2 c_0 \omega_L^2}{\varepsilon_0 \omega_B}. \tag{33}$$

This critical value corresponds to $Q \simeq 1/4$. For the time-variable acceleration the requirement takes the form $|a(t)| < a_{\text{cr}}$. If the inequality is violated, i.e., $|q'| > 2$ (but $|q'| < 2q_0^2$), the adiabatic explosive solution (21) is valid only far enough from the resonance, where the frequency shift is larger than the critical value,

$$|\tilde{\Delta}_a| \simeq |ka|^{1/2} \simeq |\omega_B a / c_0|^{1/2},$$
$$|\tilde{\Delta}_a| \gg 2\omega_L \varepsilon_0^{-1/2}. \tag{34}$$

For lower shifts, $|\tilde{\Delta}| \ll |\tilde{\Delta}_a|$, the field growth is terminated because a too high acceleration quickly brings the electrons out of the decelerating phase of the electromagnetic field.

Combining these results with the expressions (21) and (24), we obtain the maximum amplification factor, $\kappa_d \gg 1$, and the minimal shift, $|\tilde{\Delta}_d| \ll |\tilde{\Delta}_0|$, which correspond to the saturation of the field growth for a given constant acceleration (or deceleration) a:

$$\kappa_d \simeq \frac{\tilde{\Delta}_0}{\tilde{\Delta}_d}, \quad |\tilde{\Delta}_d| \simeq \left(\frac{\omega_B |a|}{c_0 + \omega_L^2/\varepsilon_0} \right)^{1/2}. \tag{35}$$

In addition to the limitations (25) and (26), it is clear that always $\kappa_d \ll (B_0 / E_0 \sqrt{\varepsilon_0})^{1/2}$.

It is convenient to introduce the delay time t_d, characterizing the growth of the field from the small initial value, $|\tilde{\mathcal{E}}_0| \simeq 2\pi |\tilde{\mathcal{J}}_0| / \varepsilon_0 |\tilde{\Delta}_0|$, to the maximum value, $|\tilde{\mathcal{E}}_d| = \kappa_d |\tilde{\mathcal{E}}_0|$:

$$t_d \simeq \frac{\tilde{\Delta}_0}{ka} \simeq \left| \frac{c_0 \tilde{\Delta}_0}{\omega_B a} \right| \gg |\tilde{\Delta}_0^{-1}|. \tag{36}$$

The last inequality excludes too high electron acceleration, which forbids the explosive amplification effect from the very beginning, for the given initial frequency shift $\tilde{\Delta}_0$.

The characteristic time of the explosive amplification process near the saturation level can be defined by the equality $2|\tilde{\mathcal{E}}(t_d - \tau)| = |\tilde{\mathcal{E}}(t_d)|$, which gives the value

$$\tau \simeq t_d / \kappa_d \simeq \tilde{\Delta}_d / ka \gtrsim |\tilde{\Delta}_d^{-1}|. \tag{37}$$

The optimal amplification regime is evidently reached for the acceleration equal to the critical value (33). It corresponds to the minimal characteristic time (37) of the order of the period of plasma oscillations, $\tau(a_{\text{cr}}) \sim \sqrt{\varepsilon_0}/\omega_L$. It does not make sense to shorten this time by choosing $a \gg a_{\text{cr}}$, since it gives the explosive factor (35) much smaller than the limiting value (24): $\kappa_d \ll \kappa_m$. Note also that in the case $a = \text{const}$ one can achieve a large explosive factor, $\kappa_d \gg 1$, only if the duration $t_d \simeq \kappa_d \tau$ of the preliminary stage of the process greatly exceeds the period of plasma oscillations. This is an undesirable feature because of the existence of competing relaxation processes and wave instabilities. As will be shown in the next section, the delay time can be reduced by increasing the electron acceleration at the preliminary stage of the amplification and keeping the value $a \sim a_{\text{cr}}$ at the final stage.

V. TIME-VARIABLE ELECTRON ACCELERATION AND THE IMPROVED QUASIADIABATIC SOLUTION

In this section we study the possibility to reduce the delay time by employing time-variable and high enough acceleration, $|a(t)| \gtrsim a_{\text{cr}}$. The acceleration does not change sign during the amplification process, while the shift (17) $q(\theta)$ in Eqs. (16) and (29) changes monotonously from a large value, $|q_0| \gg 1$ at $t=0$, to a small value, $|q_d| \lesssim 1$ at some $t = t_d$. Starting again from one normal cyclotron wave, we have to find the evolution of the ratio of the electric field to the cyclotron current, $H = if/g$. It obeys the equation of the Riccati type:

$$H' = i(H - H_c)(H + 1/H_c),$$
$$H_c \equiv (if/g)_c = q(1 - \sqrt{1 + q^{-2}}), \tag{38}$$

where H_c is the adiabatic value of H in a normal cyclotron wave, defined by Eq. (18).

We are interested in the quasiadiabatic solution to Eq. (38) with small deviation from H_c. Therefore, we may put approximately $H \simeq H_c$ in the last multiplier in the right-hand side of Eq. (38) and obtain immediately the general solution for an arbitrary function $q = q(\theta)$:

$$H - H_c \simeq -\exp\left(-2i \int_0^\theta q\sqrt{1 + q^{-2}}\, d\theta \right) \times \int_0^\theta H_c' \exp\left(2i \int_0^{\theta'} q\sqrt{1 + q^{-2}}\, d\theta'' \right) d\theta'. \tag{39}$$

Here we do not consider the discontinuity of the function $H_c(q)$ at the resonance point $q=0$, and assume that $q(\theta)$ is of fixed sign $[q(\theta)<0$ or $q(\theta)>0]$. It suffices for describing the main features of the explosive amplification.

Integration by parts shows that the correction (39) is negligible as compared with H_c if $|q'|\ll 2(q^2+1)$, i.e., for any function $a(t)$ that is smooth enough and satisfies the inequality

$$|ka(t)|\ll|\tilde{\Delta}(t)|^2+2\omega_L^2/\varepsilon_0. \tag{40}$$

Thus, we obtain the minimally possible delay time if for each instantaneous value of frequency shift, $|\tilde{\Delta}(t)|\gtrsim\omega_L\sqrt{2/\varepsilon_0}$, we choose the acceleration equal to the corresponding instantaneous critical value $a_{\mathrm{opt}}(t)$; cf. Eq. (33) for $|\tilde{\Delta}(t_d)|\lesssim\omega_L\sqrt{2/\varepsilon_0}$. This optimal choice of the electron acceleration can be described qualitatively by the model relation

$$\mu q'=-(q^2+1)\ \mathrm{sgn}(q_0), \tag{41a}$$

that is,

$$\mu ka(t)=\left(\frac{1}{2}|\tilde{\Delta}(t)|^2+\frac{\omega_L^2}{\varepsilon_0}\right)\ \mathrm{sgn}(\tilde{\Delta}_0). \tag{41b}$$

Here the model parameter $\mu\gtrsim 1$, so that close to the resonance we have $|a(t_d)|\simeq a_{\mathrm{cr}}$. For $\mu=1/2$, we obtain the result (34) that denotes the limit for the adiabatic approximation.

The exact solution to Eq. (41a), with the initial condition at zero time $q(0)=q_0$, is

$$q=\tan[\arctan(q_0)-\theta/\mu], \tag{42a}$$

or, in terms of the real time and *the optimal acceleration*,

$$ka_{\mathrm{opt}}(t)=\frac{\omega_L^2}{\varepsilon_0\mu\cos^2[\arctan(\tilde{\Delta}_0\omega_L^{-1}\sqrt{\varepsilon_0/2})-\omega_L t/\mu\sqrt{2\varepsilon_0}]}. \tag{42b}$$

It shows that, contrary to the case of a constant acceleration (36), the delay time for the optimal variant,

$$t_d\sim\mu\tau(a_{\mathrm{cr}})\sim\mu\sqrt{\varepsilon_0}/\omega_L\,,\quad \mu\gtrsim 1, \tag{43}$$

can be reduced to the several (of order μ) periods of plasma oscillations, while the amplification factor is remained to be maximally possible, of the order of (24). Note that the optimal acceleration (42b) decreases from the large initial value to the critical value (33).

In the optimal variant (42a), the Riccati equation (38) cannot be solved analytically even for $|q|\gg 1$, when $|\tilde{\Delta}|\propto(\mathrm{const}-t)^{-1}$ and $a_{\mathrm{opt}}\propto(\mathrm{const}-t)^{-2}$. The numerical solution is presented in Fig. 5 along with the quasiadiabatic solution (39), for the case $\mu=0.5$. The difference between curves $|H(\theta)|$ and $|H_c(\theta)|$ becomes smaller with increasing parameter μ. These curves demonstrate that the coherent cyclotron oscillations of gyrating electrons can be nearly completely transformed into the resonant electromagnetic field as a result of explosive amplification.

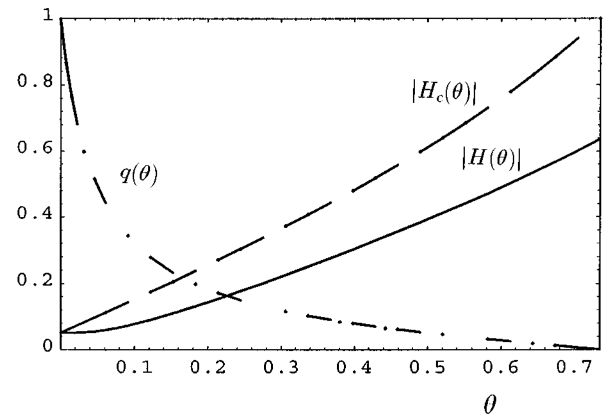

FIG. 5. Explosive amplification of a normalized electric field amplitude $H=if/g$ [see Eqs. (14)–(17)] for the optimal electron acceleration (41), and the corresponding frequency shift $q(\theta)$ (dot-dashed line): solid line is the absolute value of the solution to the Riccati equation (38), $|H(\theta)|$; dashed line is the absolute value of the adiabatic solution, $|H_c(\theta)|$. The solutions are plotted for the value of $\mu=0.5$ and for the initial conditions $q(0)=q_0=10$, $H(0)=H_c(0)$ corresponding to one normal cyclotron wave.

In general, one can vary the value of acceleration more slowly: $Gq'=-2(q^2+1)^{3/2}\mathrm{sgn}(q_0)$, where $G\gtrsim 1$. In this case the difference from the adiabatic solution is bounded by a small model parameter G^{-1}, since, according to the exact solution of Eqs. (29), the transformation factor (30) of the cyclotron wave into the electromagnetic one is much less than unity:

$$Q_{\mathrm{out}}=(G^2+1)^{-1}\sin^2\{\sqrt{G^2+1}[\arctan(q_{\mathrm{out}})-\arctan(q_0)]/2\}. \tag{44}$$

Of course, the above conclusion is valid if the function $q(\theta)$ does not change sign and, therefore, the discontinuity of the dispersion branches $\omega_{c,\mathrm{em}}(k)$ does not matter.

For large shifts, $|q|\gtrsim 2$, the function $q_{\mathrm{out}}=q(\theta)$ in Eq. (44) takes the following form:

$$|q|\simeq(q_0^{-2}-4G^{-1}\theta)^{-1/2}, \tag{45a}$$

i.e.,

$$|ka(t)|\simeq\frac{2\omega_L^2}{\varepsilon_0 G}\left(\frac{2\omega_L^2}{\varepsilon_0\tilde{\Delta}_0^2}-\frac{4\omega_L t}{G\sqrt{2\varepsilon_0}}\right)^{-3/2}\gtrsim|ka_{\mathrm{cr}}|. \tag{45b}$$

Evidently, even this rather smooth variation of electron acceleration shortens the delay time of explosive amplification up to several (of order G) periods of plasma oscillations, providing at the same time the highest possible amplification factor (24). Note that both in optimal (42b) and in slower (45a) and (45b) regimes, the acceleration of electrons (13) decreases from large initial values, correspondingly, $a(0)\simeq\tilde{\Delta}_0^2/[2\mu k\,\mathrm{sgn}(\tilde{\Delta}_0)]$ and $a(0)\simeq\tilde{\Delta}_0^3\sqrt{2\varepsilon_0}/(2Gk\omega_L)$, to the critical value (33), $a(t_d)\simeq 2\omega_L^2/[\varepsilon_0 k\ \mathrm{sgn}(\tilde{\Delta}_0)]$. The temporal dependence of this decrease is of hyperbolic type, i.e., opposite to the explosive growth of the field.

Now we can define more precisely than in Eq. (4) the condition for the electron collisions and collisionless (kinetic) relaxation to be negligible in the optimal variant (42b):

$$(kv_{T\parallel}+\nu)t_d \lesssim 1, \tag{46a}$$

or

$$\frac{v_{T\parallel}}{c}+\frac{\nu}{\omega_B\sqrt{\varepsilon_0}} \lesssim \frac{\omega_L}{\varepsilon_0\omega_B}. \tag{46b}$$

This restriction is $2\kappa_m$ times weaker than the first inequality in Eq. (4) that was used, e.g., in Eq. (25). According to the inequalities (46a) and (46b) if we violate the first inequality in Eq. (4), i.e., start with the harmonic electron oscillations placed inside the cyclotron line (see Fig. 2), the explosive amplification still takes place. However, in fact it will be less pronounced, leading to the lower value of the amplification factor κ_Δ [see Eq. (25)] since the field growth will be suppressed until the frequency of the normal cyclotron wave shifts out of the cyclotron line.

VI. RESTRICTIONS ON THE EXPLOSIVE AMPLIFICATION RATE FROM THE INSTABILITY PROCESSES AND SPATIAL INHOMOGENEITY

In addition to the electron relaxation processes, the explosive amplification phenomenon can be affected by various (kinetic) instabilities, well pronounced in the monoenergetic (helical) electron flow [4–7,28–32]. For the estimations, we will neglect the nonadiabatic effects, substituting the instantaneous (nonrelativistic) values of electron longitudinal ($\parallel \mathbf{B_0}$) velocity v and transverse (rms) velocity $u_\perp$ into the known expressions for the instability growth rates. This is possible, at least, for the X mode, since its dynamics is almost adiabatic in the regime of efficient explosive amplification. We also put $\varepsilon_0=1$ for simplicity.

The maximum growth rates, γ_m, of the most important maser (relativistic) and magnetic (cyclotron) instabilities are reached at $u_\perp/c \gtrsim \omega_L/\omega_B$ for the wave numbers far from the resonance condition (see Fig. 2) [7,30]:

$$\gamma_m \simeq \omega_L u_\perp/c\sqrt{2}; \quad u_\perp \ll c. \tag{47}$$

For the resonant wave numbers, these instabilities partially cancel each other, lowering the total growth rate of a normal cyclotron wave (10) [4,7]. For the small transverse velocities, $u_\perp/c \ll \omega_L/\omega_B$, the growth rate is also lower than (47) due to collective electron effects. Thus, the rise time of the above instabilities is at least $c/4u_\perp$ times longer than the period of plasma oscillations. Therefore, with the optimal choice of the acceleration regime, these instabilities cannot compete with the explosive amplification process, because the latter develops much faster: its characteristic time (37) is $\tau \ll \gamma_m^{-1}$. This is true even for the constant acceleration, if the latter is strong enough: $u_\perp\omega_L^2/\omega_B \ll |a| \lesssim a_{\rm cr}$. For the arbitrary constant acceleration, the explosive amplification dominates provided $t_d=\kappa_d\tau \lesssim \gamma_m^{-1}$. This inequality places the restriction on the maximum amplification factor (35):

$$\kappa_d=\frac{|\tilde\Delta_0|}{\omega_L} \lesssim \frac{|a|\omega_B}{\omega_L^2 u_\perp}, \quad |a|\lesssim a_{\rm cr},$$

$$\kappa_d=\left|\frac{\tilde\Delta_0^2 c}{\omega_B a}\right|^{1/2} \lesssim \frac{c}{u_\perp}\left|\frac{4a}{a_{\rm cr}}\right|^{1/2}, \quad |a|\gtrsim a_{\rm cr}. \tag{48}$$

For the critical acceleration (33), the amplification factor cannot be much larger than $c/u_\perp$.

The above restriction is valid in the case of a nonzero dispersion of longitudinal electron velocities $v_{T\parallel}$, because this dispersion does not affect significantly the maser instability, though it suppresses the cyclotron instability for $v_{T\parallel}/u_\perp \gtrsim (u_\perp/c)^{5/2}(\omega_L/\omega_B\sqrt{2})^{1/2}$. The dispersion of transverse electron velocities suppresses the maser instability, while the explosive amplification does not depend on the transverse velocity distribution. The cyclotron instability in this case takes the form of the Weibel instability [30–32] that has the growth rate not exceeding the value (47) and also cannot affect the explosive amplification.

If the electron beam has a nonzero velocity $v_\parallel$ with respect to ions, the Buneman instability of longitudinal plasma waves may develop [7,32]. The maximum growth rate of the instability is reached for the plasma wave numbers $k_\parallel \sim \omega_L/v_\parallel$ and is equal to

$$\gamma_\parallel \simeq \frac{\sqrt{3}}{2^{1/3}}\left(\frac{m}{2M}\right)^{1/3}\omega_L, \tag{49}$$

where M is ion mass. Due to the presence of a small parameter, $(m/2M)^{1/3}$, in Eq. (49) [cf. Eq. (47)], the Buneman instability also cannot compete with the explosive amplification.

Now we turn to the restriction on the width of the temporal, $\Delta\omega$, and spatial, Δk, spectrum of *a cyclotron wave envelope*. This would allow one to consider the explosive amplification of field in the nonstationary (with the temporal scale $T=|\tilde\Delta|/|d\tilde\Delta/dt|$) and inhomogeneous (with the spatial scale L) electron beam and external quasistatic fields, $\mathbf{E_0}$ and $\mathbf{B_0}$. The frequency bandwidth $\Delta\omega$ of a well-defined envelope evidently cannot be less than $1/T$, but at the same time should not exceed the frequency interval of cyclotron waves (10), where their amplification factor is approximately constant ($\kappa \simeq |\tilde\Delta|\sqrt{\varepsilon_0}/\omega_L$):

$$\frac{1}{t_d} \lesssim \frac{1}{T} \lesssim \Delta\omega \lesssim \frac{\omega_L^3}{(2\varepsilon_0)^{3/2}\tilde\Delta^2} \ll |\tilde\delta_c|. \tag{50}$$

Let us estimate the value Δk, i.e., the allowable longitudinal spread of an envelope, $1/\Delta k$. Consider the most interesting case, when the average velocity of an electron beam, v_b, is less than the typical group velocity of cyclotron waves, $v_c \equiv \partial\omega_c/\partial k$, along the magnetic field:

$$v_b \simeq \frac{1}{2}\left(v_0+\frac{\tilde\Delta_0}{k}\right) \lesssim v_c \simeq v_b+\frac{c_0\omega_L^2}{2\varepsilon_0\tilde\Delta^2} \lesssim c_0/2. \tag{51}$$

Then the propagation path of a beam, $l_b=t_d v_b$, does not exceed that of a wave envelope, $l_c=t_d v_c$, during the delay time t_d:

$$l_b \lesssim l_c \simeq \frac{t_d c_0 \omega_L^2}{2\varepsilon_0 \tilde{\Delta}_0^2} = \begin{cases} \dfrac{c_0^2 \omega_L^2}{2\varepsilon_0 \omega_B |\tilde{\Delta}_0 a|}, & a=\text{const} \\ \dfrac{\mu c_0 \omega_L}{2\sqrt{\varepsilon_0}\tilde{\Delta}_0^2}, & a=a_{\text{opt}}(t). \end{cases} \tag{52}$$

The explosive amplification process now depends on the finite scales of inhomogeneity of a beam and external fields, L, and of an envelope, $1/\Delta k$, but this dependence does not eliminate the phenomenon itself. The analysis can be carried out by means of *spatiotemporal geometric optics* [25–27], if the following inequalities are satisfied:

$$\frac{1}{l_c} \lesssim \frac{1}{L} \lesssim \Delta k \lesssim \frac{\omega_L}{2\pi c}. \tag{53}$$

They are consistent with the inequalities (50) due to the relation $\Delta\omega \sim v_c \Delta k$. The last inequality in (53) takes into account the characteristic (plasma) dispersion scale of two curves (9) of the X-mode (Fig. 2). It allows one to introduce correctly the spectral notion of a cyclotron wave envelope. Such a quasimonochromatic wave envelope is well defined both in space-time (z,t), where its ray path is defined by the group velocity $\partial\omega_{1,2}/\partial k$, and in phase space (k,ω), where it can be described by a point moving along the local dispersion curve. As a result, the inhomogeneous and nonstationary problem is reduced to the purely nonstationary problem. The latter has been already solved for the homogeneous and boundless plane wave. We can use the obtained results after the substitution of local parameters of electron beam and external fields along the ray path of a wave envelope.

Thus, the decrease (13) of a local frequency shift, $\tilde{\Delta}(z,t)$, along the envelope path $z=z_c(t)$, necessary for the explosive amplification, and the corresponding sufficiently strong "acceleration," $a(t)=-k^{-1}d\tilde{\Delta}(z_c(t),t)/dt$, can be provided not only by time variability, but also by inhomogeneity of external conditions [e.g., of the magnetic field $\mathbf{B_0}(z)$]. This provides an additional opportunity for the optimal control of the amplification process. The schematic examples are presented and explained in Fig. 6, where a possible drift of wave numbers is taken into account along with a frequency drift shown earlier in Fig. 3.

It is relevant here to point out the relation of the above nonstationary problem with the well-known problem on the escape of the X mode from a steady-state magnetized plasma [33–36]. Both problems deal with the transformation of a partial cyclotron wave into a partial electromagnetic wave. Due to the stationary formulation of the problem, the explosive amplification effect was not revealed there. Of course, the effect of explosive amplification also cannot provide the penetration of a wave envelope through the cyclotron resonance.

VII. ESTIMATIONS AND LIMITS FOR THE REALISTIC CONDITIONS

In this section we give the numerical estimates of explosive amplification parameters, assuming $\varepsilon_0=1$, small initial longitudinal velocity of a beam, $|v_0| \ll c_0|\tilde{\Delta}_0|/\omega_B$, and transverse (rms) electron velocity less than or of the order of

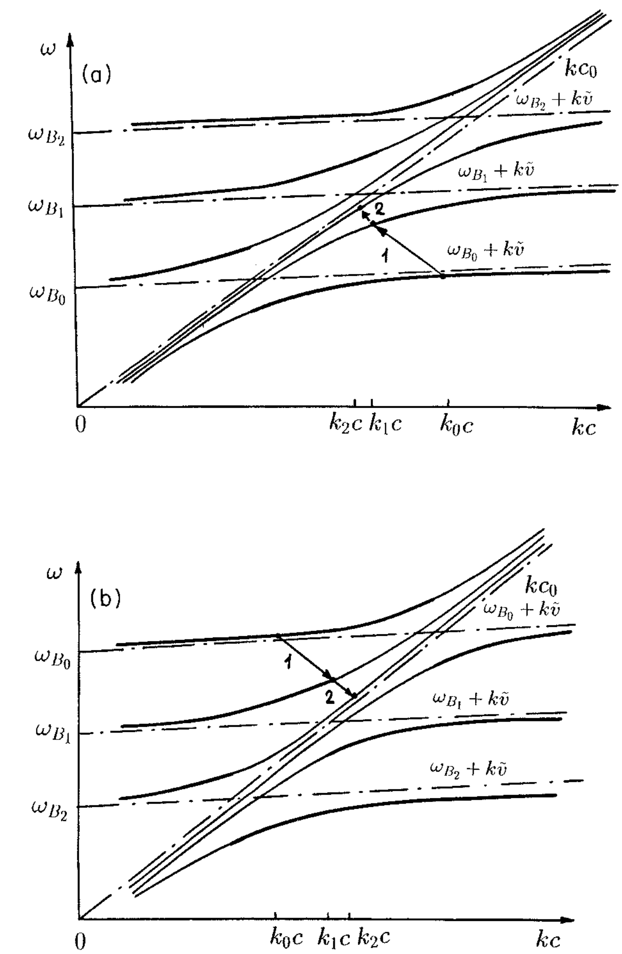

FIG. 6. Variants of the adiabatic displacement of a wave envelope along the variable cyclotron (thick line) and electromagnetic (thin line) dispersion branches of the X mode in the electron flow moving along the inhomogeneous magnetic field with a constant velocity $\tilde{v}$ (cf. Fig. 3). The cases (a) and (b) correspond to the magnetic field increasing and decreasing along the path of an envelope. Steps 1 correspond to explosive amplification of electromagnetic field in the envelope of the cyclotron waves. At the steps 2 the envelopes are transformed into that of the electromagnetic waves with nearly constant electromagnetic field.

$u_\perp \lesssim 10^{-5}c$. Consider two typical values of dc electric field, $E_0'=10^{-2}E_0^{\text{cr}}$ and $E_0''=E_0^{\text{cr}}$, corresponding to the accelerations $a'=10^{-2}a_{\text{cr}}$ and $a''=a_{\text{cr}}$. According to Eqs. (48) and (33), these values provide large enough amplification factors, $\kappa'=10^2$ and $\kappa''=10^4$, with corresponding initial frequency shifts $|\tilde{\Delta}_0'|=10^2\omega_L$ and $|\tilde{\Delta}_0''|=\sqrt{3}\times10^4\omega_L$. The critical accelerating field is defined by

$$E_0^{\text{cr}} \equiv \frac{m a_{\text{cr}}}{e} = \frac{8\pi N m c^2}{\varepsilon_0^{3/2} B_0} \simeq \frac{2\times10^{-5}N}{B_0}, \tag{54}$$

and all estimations are given in cgs units.

The initial energy density W_0 of a cyclotron wave is assumed to be small enough,

$$W_0 \simeq \frac{\kappa^2|\tilde{\mathscr{E}}_0|^2}{4\pi} \sim \frac{(\omega_L/2\omega_B)^3 B_0^2}{4\pi\kappa} \lesssim \frac{B_0^2}{4\pi(8\kappa^2)^2}, \tag{55}$$

according to Eq. (26). Choose the plasma frequency to have the highest possible value,

$$\omega_L^{\prime,\prime\prime} \simeq \omega_B/2\kappa^{\prime,\prime\prime}, \quad \text{i.e.,} \quad N^{\prime,\prime\prime} \simeq 2\times 10^4 (B_0/\kappa^{\prime,\prime\prime})^2, \tag{56}$$

in order to satisfy the first inequality in Eq. (4) with $\widetilde{\delta}_c \simeq -\omega_L^2/(2\varepsilon_0 \widetilde{\Delta}_0) \sim \omega_L/\kappa$, and also to obtain the maximum final field amplitude, $|\widetilde{\mathscr{E}}_{\text{out}}| \sim B_0/30\kappa^2$, in accordance with Eqs. (27) and (55). Such an amplitude is reached close to the cyclotron resonance, $|\widetilde{\Delta}_{\text{out}}(t_d)| \lesssim \omega_L$, after the delay time $t_d \sim c/2a$ (namely, $t_d' = 100\kappa'/2\omega_L'$ and $t_d{}'' = \sqrt{3}\kappa''/2\omega_L''$), when the electron beam achieves a moderately relativistic velocity $v_d = c_0|\widetilde{\Delta}_0|/\omega_B \sim c/2$ and propagates the distance $l_b \sim c^2/4a$ of the order of the propagation path (52), $l_c \simeq t_d v_c$, of a wave envelope.

The above delay times are in both cases (56) of the order of one thousand periods of plasma oscillations. Note that for the parameters (56), the value (54) of E_0^{cr} is $B_0/2\kappa^2$ and the critical acceleraton (33) is $a_{\text{cr}} \simeq c\omega_B/2\kappa^2$. Therefore, the value of a' unexpectedly turns out to be 100 times larger than a''. Note also that for the explosive amplification mechanism the accelerating dc electric field can be replaced by the time-variable gyrofrequency

$$a \leftrightarrow c_0 \omega_B^{-1} d\omega_B/dt, \quad \text{i.e.,} \quad E_0 \leftrightarrow \omega_B^{-1} dB_0/dt. \tag{57}$$

This "time variability" may be also due to the inhomogeneity of a steady magnetic field along the beam propagation path. In this case the "acceleration" a is determined by the local inhomogeneity scale of a magnetic field, L_B, and by the variable wave velocity v_c:

$$L_B \equiv B_0 |dB_0/dz|^{-1} = v_c c_0/a. \tag{58}$$

Now we list the numerical estimations in two cases (56) for the magnetic field $B_0 = 10^5 G$, attainable in laboratory plasma and typical for astrophysical situations, with maximum initial shift $|\widetilde{\Delta}_0^{\prime,\prime\prime}| \sim \omega_B/2 \simeq 10^{12}\text{s}^{-1}$ and the wavelength $\lambda = 2\pi c/\omega_B \simeq 0.1$ cm. For $E_0' \simeq 5\times 10^{-7} B_0 = 15$ V cm^{-1} we have

$$v_{T\|}' \lesssim 3\times 10^{-5} c, \quad N' = 2\times 10^{10} \ \text{cm}^{-3},$$
$$W_0' = |\widetilde{\mathscr{E}}_{\text{out}}|^2/4\pi \sim 0.1 \ \text{erg cm}^{-3}, \tag{59}$$
$$\omega_L' \simeq 10^{10} \ \text{s}^{-1}, \quad t_d' \sim 0.5 \ \mu\text{s}, \quad l_b' \sim 40 \ \text{m}.$$

For $E_0'' \simeq 5\times 10^{-9} B_0 = 0.15$ V cm^{-1} we obtain

$$v_{T\|}'' \lesssim 3\times 10^{-9} c, \quad N'' = 2\times 10^{6} \ \text{cm}^{-3},$$
$$W_0'' = |\widetilde{\mathscr{E}}_{\text{out}}|^2/4\pi \sim 10^{-9} \ \text{erg cm}^{-3}, \tag{60}$$
$$\omega_L'' \simeq 10^{8} \ \text{s}^{-1}, \quad t_d'' \sim 90 \ \mu\text{s}, \quad l_b'' \sim 7 \ \text{km}.$$

Thus, the conditions for the explosive amplification seem to be quite attainable. Moreover, we can relax the restrictions imposed on the dispersion of the longitudinal electron velocity $v_{T\|}$, and reduce the delay time, $t_d \simeq |\widetilde{\Delta}_0| c_0/(\omega_B a)$, the final velocity, $v_d \simeq c_0|\widetilde{\Delta}_0|/\omega_B$, and the length of a beam, $l_b = t_d v_d/2$, at the expense of the amplification factor, $\kappa \simeq |\widetilde{\Delta}_0|/\omega_L$.

VIII. CONCLUSIONS

In conclusion, we have proved the possibility of the explosive amplification of a coherent electromagnetic field at the expense of the wave energy of phased (coherent) cyclotron oscillations of electrons, accelerated along the external magnetic field. The amplification develops due to the rapid quasiadiabatic change of the ratio of electron to field energy in a self-consistent cyclotron wave close to the cyclotron resonance, when the (Doppler-shifted) electron gyrofrequency is tuned to the partial frequency of the electromagnetic wave. Due to the energy conservation, the amplification of a field amplitude is accompanied by the decreasing of an amplitude of a coherent transverse current. In this aspect, the proposed nonstationary mechanism of the explosive amplification principally differs from the known mechanism of the exponential amplification in, e.g., maser or cyclotron instability, when the field and current amplitudes are growing simultaneously, at the expense of initial energy of incoherent oscillations in a system of nonlinear oscillators.

Note also that the proposed mechanism of field amplification does not require the existence of any other wave (waveguide or resonator eigenmode, the normal wave of a background plasma, etc. [1,37]) that would be in resonance with the preliminary excited cyclotron wave of a beam. In our case, it is the electric field of the initial cyclotron wave of a beam that is efficiently amplified. Therefore, the situation is different from that usually discussed in electronics, in which the transfer of energy of electron oscillations to the electromagnetic field energy occurs as a result of interaction between the normal wave of a beam and some other electromagnetic mode.

It is evident that the explosive amplification effect is common for an ensemble of arbitrary harmonic oscillators, not only of the cyclotron ones. The required shift of the oscillator frequency with respect to the frequency of electromagnetic wave can be related not only with the Doppler effect in accelerating fields, but also with other external forces that directly change the frequency of motionless oscillators, like in Zeeman or Stark effects. For the applications, it is important that, due to the resonant nature of the phenomenon, the strong amplification originates from a very small shift of the oscillation frequency. Moreover, we have shown that the required frequency shift can be carried out so rapidly that the explosive amplification rate greatly exceeds the rates of the known mechanisms of the field amplification. This can lead to new physical effects that have not been widely discussed before.

At the same time, it should be emphasized that the explosive amplification can be most effectively employed only at the final stage of the generation of a high-intensity electromagnetic field, when the coherent wave of phased electron oscillations is already prepared by a preliminary pumping. Then the ''explosive'' mechanism provides an extremely quick and efficient transfer of energy of these oscillations to the energy of coherent electromagnetic field which can freely escape from an ensemble of active oscillators shifted out of the resonance.

We hope that the obtained results may have both laboratory applications in the physics of magnetized electron beams, and astrophysical applications in the theory of nonsteady processes in magnetospheres, jets and accretion disks of degenerate stars and black holes.

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